Geometric Approach to a Dilation Theorem

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ABSTRACT

Problem: Given operators $A_j \geqslant 0$ on Hilbert space $\mathcal{H}$, with $\Sigma A_j = 1$, to find commuting projectors E_j on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that (for all j) $x^*A_jy = x^*E_jy$ for $x, y \in \mathcal{H}$. This paper gives an explicit construction, quite different from the familiar solution.

INTRODUCTION

Given k commuting Hermitians $E_0, E_1, \ldots, E_{k-1}$ on a Hilbert space $\mathcal{K}$, we may regard them as sesquilinear forms having simultaneous spectral resolution—in the finite-dimensional case, having a common system of principal axes. For a proper subspace $\mathcal{H}$ of $\mathcal{K}$, let A_j be the sesquilinear form defined by $x^*A_jy = x^*E_jy$ $(x, y \in \mathcal{H})$, the *compression* of E_j to $\mathcal{H}$; then the A_j are again Hermitian, but need not commute. Thus, for example,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{commutes with} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix},$$

but if we choose $\mathcal{H}$ to be the set of all

$$\begin{bmatrix} \xi \\ \eta \\ 0 \end{bmatrix},$$

then the compressions to $\mathcal{H}$ have matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and do not commute. Two ellipsoids with common principal axes may in some plane have elliptical sections which fail to have common principal axes.

This raises the inverse problem: Given Hermitian $A_0, A_1, \ldots, A_{k-1}$ on $\mathcal{H}$, to construct commuting $E_0, E_1, \ldots, E_{k-1}$ on $\mathcal{K}$ such that each A_j is the compression of E_j or (equivalent terminology) E_j is a *dilation* of A_j to $\mathcal{K}$. In discussing this, we may assume without loss of generality that all $A_j \geq 0$. Then we may also assume that $A_0 + A_1 + \cdots + A_{k-1} = 1$, the identity. For we can make the sum ≤ 1 simply by multiplying by a constant, and if this does not give us equality we can, by increasing k by 1, throw in another A_j to produce the desired sum.

This dilation problem has a well-known answer:

THEOREM 1. *Assume that A_j $(j = 0, 1, \ldots, k-1)$ are operators on $\mathcal{H}$ with $A_j \geq 0$, $\Sigma_0^{k-1} A_j = 1$. Then we can construct a space $\mathcal{K}$, an imbedding ι of $\mathcal{H}$ into $\mathcal{K}$, and operators E_j on $\mathcal{K}$, such that*

(i) $E_j \geq 0$, $\Sigma_0^{k-1} E_j = 1$,
(ii) *E_j is a dilation of A_j to $\mathcal{K}$ for all j, that is, $A_j = \iota^* E_j \iota$, and*
(iii) *$E_i E_j = 0$ for $i \neq j$.*

The E_j not only commute, they form a complete system of orthoprojectors.

This result is usually stated in more general form:

THEOREM 2. (Naĭmark's dilation theorem [5, Theorem I.8.2]). *Let $A(\cdot)$ be a function on the interval $[a,b]$ whose values are operators on a Hilbert space $\mathcal{H}$ with $A(a) = 0$, $A(b) = 1$, and $A(t) \leq A(u)$ for $t < u$. Then we can construct a Hilbert space $\mathcal{K}$, an imbedding ι of $\mathcal{H}$ into $\mathcal{K}$, and a function $E(\cdot)$ on $[a,b]$ whose values are operators on $\mathcal{K}$, such that*

(i) *$E(a) = 0$, $E(1) = 1$, and $E(t) \leq E(u)$ for $t < u$,*
(ii) *$E(t)$ is a dilation of $A(t)$ to $\mathcal{K}$ for all t, that is, $A(t) = \iota^* E(t) \iota$, and*
(iii) *$E(t)E(u) = E(t)$ for $t \leq u$.*

The $E(t)$ not only commute, they form a spectral measure (resolution of the identity). Note that technically I should normalize by insisting that (say) $\lim_{t \to u^-} A(t) = A(u)$ and likewise for $E(\cdot)$. This technicality will be ignored in what follows.

(In both theorems we may identify $\mathcal{H}$ with $\iota\mathcal{H}$ on many occasions—as I did above.)

Theorem 1 is obtained from Theorem 2 as the finite-dimensional spectral theorem is obtained from the infinite-dimensional. Namely, given $A_0,\ldots,A_{k-1}$ as in Theorem 1, we define $A(\cdot)$ on $[0,k]$ by $A(t)=\Sigma_{j<t}A_j$; and then, having got $E(\cdot)$ of Theorem 2, we find we can return to operators $E_0,\ldots,E_{k-1}$ by reversing this.

As an answer to the question with which we began, Naĭmark's theorem is overkill. Starting with a few modest operators in 2-space, it yields an apparently infinite chain of orthoprojectors in an apparently infinite-dimensional space. Although the excess baggage may then be laid aside, one might wish never to have taken it on. This is one of the advantages of the present approach.

I begin in Sec. 1 with a self-contained proof of Theorem 1, based on an old idea by E. Michael [3, Theorem 4; 1, Sec. 5]. An earlier treatment of Theorem 1, in the special case that each A_j has rank 1, is by H. Hadwiger ([2, Satz I]; cf. [4]). I hope my argument rivals the geometric naturality of Hadwiger's.

Just as any numerical-valued measure may be specified by approximating it by piecewise constant functions, so may any spectral measure. Thus the present proof of Theorem 1 should afford a new proof of Naĭmark's dilation theorem. This possibility, referred to in [1, Sec. 5], is explained in Sec. 2 below. For technical reasons it is easier to start afresh, so that Sec. 2 is nearly independent of Sec. 1. Section 2 may be regarded as providing the matricial approach to Naĭmark's theorem, as J. J. Schaffer's proof [5, Sec. I.5] does for Sz.-Nagy's dilation theorem [5, Sec. I.4].

1. THE CONSTRUCTION FOR FINITE SUMS

The basic idea of Michael applies to a single operator. In that case we are given on $\mathcal{H}$ only a positive operator A_0, such that $A_1=1-A_0$ is also positive. These already commute; the only call for us to step outside of $\mathcal{H}$ is in order that we may achieve $E_0E_1=0$.

Now each A_j, being positive, has a positive square root F_j, and these commute. The desired solution is

$$E_0=\begin{pmatrix} F_0F_0 & F_0F_1 \\ F_0F_1 & F_1F_1 \end{pmatrix}, \qquad E_1=\begin{pmatrix} F_1F_1 & -F_0F_1 \\ -F_0F_1 & F_0F_0 \end{pmatrix}$$

on the direct sum of $\mathcal{H}$ with itself. Indeed, these are seen (using commutativity of the F_j) to be Hermitian and to have product 0. The compression of E_j to the first coordinate subspace (copy of $\mathcal{H}$) is $F_j^2 = A_j$ as desired. And their sum is

$$E_0 + E_1 = \begin{pmatrix} A_0 + A_1 & 0 \\ 0 & A_0 + A_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the identity on the new space. So much for the case $k=2$.

This neat picture inevitably gets messier as soon as $k>2$, just because the A_j fail to commute. Let us look first at the first appearance of non-commuting, for $k=4$: A_0, A_1, A_2, A_3. The above construction can be applied to the two operators $A_0 + A_1$ and $A_2 + A_3$, for they are positive operators adding to 1. Let F_0 and F_1 respectively denote their positive square roots. Then we are led to expect $E_0 + E_1$ to be

$$\begin{pmatrix} F_0F_0 & F_0F_1 \\ F_0F_1 & F_1F_1 \end{pmatrix}$$

or some direct sum of copies of this.

To proceed, we would like next to treat A_0 and A_1 as in the case $k=2$; unfortunately, their sum is not 1 but $A_0 + A_1$, with which they need not commute. We recall however the familiar fact

LEMMA. *If $0 \leqslant A \leqslant BB^*$, then there exists C such that $0 \leqslant C \leqslant 1$ and $A = BCB^*$.*

This allows us to write $A_0 = F_0F_{00}F_{00}F_0$, $A_1 = F_0F_{01}F_{01}F_0$, and hope to apply the previous considerations to dilate the $F_{0\nu}$ $(\nu = 0,1)$—which do add to 1—while "shielding" them by the F_0 from all things with which they do not commute. That is what will be done.

For the general case, let $k = 2^m$. (Should we be given $2^{m-1} < k < 2^m$, we throw in a few more A_j which are zero.) The construction is in m tiers, so let us write subscripts in binary notation: when $i,j,l \in \{0,1,\ldots,2^m - 1\}$, write

$$i = \varepsilon_1\varepsilon_2 \cdots \varepsilon_m, \qquad j = \phi_1\phi_2 \cdots \phi_m, \qquad l = \psi_1\psi_2 \cdots \psi_m, \tag{1}$$

where each digit ε_κ, ϕ_κ, or ψ_κ may be 0 or 1.

STEP 1. *We may write the A_i as products of certain operators F_s (the s being strings of zeros and ones) as follows:*

$$A_i = F_{\varepsilon_1}F_{\varepsilon_1\varepsilon_2} \cdots F_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m}F_{\varepsilon_1\ldots\varepsilon_m} \cdots F_{\varepsilon_1\varepsilon_2}F_{\varepsilon_1}.$$

Another way to express this is

$$A_i = F_{i_1}F_{i_2}\cdots F_{i_{m-1}}F_iF_iF_{i_{m-1}}\cdots F_{i_2}F_{i_1},$$

by letting i_κ denote the initial segment of length κ of the string $\varepsilon_1\varepsilon_2\ldots\varepsilon_m$ which represents i. The operators $F_{\ldots}$ are positive and satisfy $F_{s0}^2+F_{s1}^2=1$ (even if s is the void string).

The proof of Step 1 is by induction on m. The case $m=1$ we started with: two positive operators adding to 1 may be written as F_0^2 and F_1^2 for positive F_0, F_1. Now assume it known that the process can be carried out up to $m-1$. Given $A_0, A_1, \ldots, A_{2^m-1}$, define $A_0^-, A_1^-, \ldots, A_{2^{m-1}-1}^-$ by $A_h^- = A_{h0} + A_{h1}$. Then $A_h^- \geqslant 0$ and $\Sigma_h A_h^- = \Sigma_i A_i = 1$. By the inductive hypothesis, choose F_s, for strings of all lengths $< m$, satisfying the conditions

$$A_h^- = F_{h_1}F_{h_2}\cdots F_{h_{m-2}}F_hF_hF_{h_{m-2}}\cdots F_{h_2}F_{h_1},$$

$F_s \geqslant 0$, $F_{s0}^2 + F_{s1}^2 = 1$. Now $0 \leqslant A_i \leqslant A_{i_{m-1}}^-$, and the Lemma applies. We choose F_i as the positive square root of the operator whose existence is asserted by the Lemma. All the required properties are immediately verifiable.

In the Lemma, C was not uniquely determined on null(B). The consequence in Step 1 is that the $F_{s\nu}$ are not determined on null(F_s). Let us require null$(F_{s0}) \supseteq$ null(F_s), say; then F_{s1} must be 1 on null(F_s).

Step 2 (Definition of the dilation). *Let $\mathcal{K}$ be the direct sum of k copies of $\mathcal{H}$, realized as the space of column vectors $(x_j)_{j=0}^{k-1}$ with each $x_j \in \mathcal{H}$. Imbed $\mathcal{H}$ into $\mathcal{K}$ by mapping each $x \in \mathcal{H}$ to $\iota x = (\delta_{j0}x)_{j=0}^{k-1}$. Define linear operators B_i from $\mathcal{K}$ to $\mathcal{H}$ by $B_i = (B_{ij})_{j=0}^{k-1}$, where, with the notation* (1),

$$B_{ij} = (-1)^{\varepsilon_1\phi_1 + \cdots + \varepsilon_m\phi_m} F_{\varepsilon_1+\phi_1}\cdots F_{i_{\kappa-1},\varepsilon_\kappa+\phi_\kappa}\cdots F_{i_{m-1},\varepsilon_m+\phi_m}$$

(the addition in certain subscript digits is of course to be understood modulo 2). Finally, the operator E_i on $\mathcal{K}$ will be defined as $B_iB_i^*$.

It is immediate that $E_i \geqslant 0$. To see that the compression $\iota^*E_i\iota$ is A_i we need to check that $B_{i0}B_{i0}^* = A_i$; this is easy from Steps 2 and 1.

Step 3. $\Sigma_j B_{ij}^* B_{lj} = \delta_{il}$.

Proof, using again the notation (1): The sum over j is an m-fold sum over the ϕ_κ, each running from 0 to 1; let us sum first over ϕ_1.

$$\sum_{\phi_1} B_{ij}^* B_{lj} = (-1)^{(\varepsilon_2+\psi_2)\phi_2 + \cdots + (\varepsilon_m+\psi_m)\phi_m} MSN,$$

where M and N are products of certain F_s, and

$$\begin{aligned} S &= \sum_{\phi_1} (-1)^{(\varepsilon_1+\psi_1)\phi_1} F_{\varepsilon_1+\phi_1} F_{\psi_1+\phi_1} \\ &= F_{\varepsilon_1} F_{\psi_1} + (-1)^{\varepsilon_1+\psi_1} F_{\varepsilon_1+1} F_{\psi_1+1} \\ &= \begin{cases} 1 & (\varepsilon_1 = \psi_1) \\ 0 & (\varepsilon_1 \neq \psi_1) \end{cases} \end{aligned}$$

because the F_ν commute and their squares add to 1. Continue, summing successively over $\phi_2, \ldots, \phi_m$. If $i = l$, then at each stage $\varepsilon_\kappa = \psi_\kappa$, so the result is 1. If $i \neq l$, then at some stage $\varepsilon_\kappa \neq \psi_\kappa$, so the result is 0 (for every value of $\phi_{\kappa+1}, \ldots, \phi_m$). This concludes Step 3.

STEP 4. $\sum_i B_{ij} B_{il}^* = \delta_{jl}$.

The proof is just like the preceding except that one sums first over ϕ_m and proceeds backward.

Now the remaining properties required of the E_i are easy to establish. First, we require $\sum_i E_i = 1$. Each operator here is a $k \times k$ block matrix; in matrix form this requirement reads

$$\delta_{jl} = \sum_i (E_i)_{jl} = \sum_i B_{ij} B_{il}^*,$$

which holds by Step 4. Finally, we require $E_i E_l = \delta_{il} E_i$, or $B_i B_i^* B_l B_l^* = \delta_{il} B_i B_l^*$. For this it is sufficient that

$$\delta_{il} = B_i^* B_l = \sum_j B_{ij}^* B_{lj},$$

which holds by Step 3. This completes the proof of Theorem 1.

It is customary to note, in presenting Naĭmark's dilation theorem, that the dilation space will be essentially unique if it is required to be minimal.

Minimality in such a context means that $\mathcal{K}$ is spanned by the images of $\iota\mathcal{H}$ under all operators which enter. For Theorem 1, that is especially simple: it is enough that $E_0\iota\mathcal{H}+E_1\iota\mathcal{H}+\cdots+E_{k-1}\iota\mathcal{H}$ be all of $\mathcal{K}$—or at least that it be dense in $\mathcal{K}$ (if $\mathcal{K}$ is infinite-dimensional so that the distinction is significant). The construction in this section gives a non-minimal $\mathcal{K}$ whenever any A_i has non-zero null space; in the finite-dimensional case this is the only way it can give non-minimal $\mathcal{K}$, but there are other ways if $\dim\mathcal{H}$ is infinite. Here are some of the details.

PROPOSITION. *Let x be a non-zero element of* null(A_i); *let κ be that index $\in\{1,\dots,m\}$ such that $F_{i_{\kappa-1}}\cdots F_{i_2}F_{i_1}x$ is a non-zero element y of null(F_{i_κ}); and let l be $i_\kappa 0\dots 0$. Then $B_l y$ is a non-zero element of $\mathcal{K}$ orthogonal to every $E_j\iota\mathcal{H}$.*

Proof. One conclusion is evident: because range(B_l) is orthogonal to range(E_j) for $j\neq l$, surely $B_l y$ is orthogonal to $E_j\iota\mathcal{H}$ for $j\neq l$. We do have to check that $B_l y$ is orthogonal to $E_l\iota\mathcal{H}$, and this is the same as proving it orthogonal to $\iota\mathcal{H}$.

Now we know

$$\begin{aligned} &y\in\mathrm{null}(F_{i_\kappa})\subseteq\mathrm{null}(F_{i_\kappa 0})\subseteq\cdots\subseteq\mathrm{null}(F_{i_\kappa 0\dots 0}),\\ &y=F_{i_{\kappa-1},\varepsilon_\kappa+1}\,y=F_{i_\kappa 1}\,y=F_{i_\kappa 01}\,y=\cdots=F_{i_\kappa 0\dots 01}\,y, \end{aligned}\tag{2}$$

by the convention which followed Step 1.

Consider $B_l y$ in the light of these relations, the definition of the components of B_l by Step 2, and the choice of l being used. On the one hand, $B_l y$ is orthogonal to $\iota\mathcal{H}$ because $B_{l0}\,y=F_{l_1}\cdots F_{l_m}\,y$ and $F_{l_m}\,y=F_{i_\kappa 0\dots 0}\,y=0$. On the other hand, we will see that $B_l y\neq 0$ by looking at another particular component. Choose j to be a string of $\kappa-1$ zeros followed by all the rest ones. Then, by (2),

$$\begin{aligned} B_{lj}y&=\pm F_{l_1}\cdots F_{l_{\kappa-1}}F_{l_{\kappa-1},\psi_\kappa+1}\cdots F_{l_{m-1},\psi_m+1}\,y\\ &=\pm F_{i_1}\cdots F_{i_{\kappa-1}}F_{i_{\kappa-1},\varepsilon_\kappa+1}F_{i_\kappa 1}F_{i_\kappa 01}\cdots F_{i_\kappa 0\cdots 01}\,y\\ &=\pm F_{i_1}\cdots F_{i_{\kappa-1}}\,y. \end{aligned}$$

But this clearly has non-zero inner product with the originally given x. ■

In the finite-dimensional case, one can ask for the number of dimensions of the minimal dilation space. The answer is elementary: $\Sigma_j \operatorname{rank}(A_j)$ dimensions are required. Indeed, $\operatorname{rank}(A_j) = \operatorname{rank}(\iota^* E_j \iota) \leqslant \operatorname{rank}(E_j)$ and $\dim \mathcal{K} = \Sigma_j \operatorname{rank}(E_j)$, so we will need at least that many dimensions. For the converse, what is to be excluded is minimality of any dilation space in which $\operatorname{rank}(E_i) > \operatorname{rank}(A_i)$ for any i. But $\operatorname{rank}(A_i)$ is the dimensionality of the image of $E_i \mathcal{K}$ under a projection, so the strict inequality would mean this projection was not one-one. Any non-zero null vector of it would contradict minimality of $\mathcal{K}$, by an argument used in the proof of the Proposition.

2. THE CONSTRUCTION FOR CONTINUOUS FAMILIES

This section concerns the proof of Theorem 2 by a slight strenthening of the construction used before. The idea of successive duplication is perhaps more natural in the present context. Ordinarily $\mathcal{K}$ will have to be infinite-dimensional even if $\dim \mathcal{H}$ is finite—even 1.

Assume therefore that, for each t with $0 \leqslant t \leqslant 1$, we are given an operator $A(t)$ on $\mathcal{H}$; that $A(0) = 0$ and $A(1) = 1$; and that $A(t) \leqslant A(u)$ whenever $t < u$.

In giving the construction, I will reuse some ideas and notation from Sec. 1. Strings $i = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_m$ of zeros and ones will again appear, but now they will sometimes represent dyadic rationals: the rational $.\varepsilon_1 \ldots \varepsilon_m$ will be abbreviated as $.i$. Thus $.i = .i0$, although i and $i0$ are distinct strings having unequal lengths m and $m+1$; in particular, if $\varnothing$ denotes the empty string of length 0, then $.\varnothing$ means 0.

STEP 1. *We may write the $A(.i)$ in terms of certain operators F_s so that*

$$A(.i + 2^{-m}) - A(.i) = F_{\varepsilon_1} F_{\varepsilon_1 \varepsilon_2} \cdots F_{\varepsilon_1 \ldots \varepsilon_m} F_{\varepsilon_1 \ldots \varepsilon_m} \cdots F_{\varepsilon_1 \varepsilon_2} F_{\varepsilon_1}. \tag{3}$$

The operators $F_{\ldots}$ are positive, and $F_{s0}^2 + F_{s1}^2 = 1$.

Evidently if a string i consists entirely of zeros (or is void), then we know $A(.i)$ ahead of time, by $A(.i) = A(0) = 0$. In expressing $A(t)$ (for all dyadic rational $t \in [0,1[$) in terms of the F_s, we can ignore strings ending in 0, because $.i0 = .i$. Now we can begin the choice of the $F_{\ldots}$ by taking F_0 to be the positive square root of $A(.1)$ and F_1 that of $1 - A(.1)$; (3) then holds for $m = 1$.

Make the inductive hypothesis that the F_s have been defined for all strings of length $\leqslant m$, with the asserted properties. Use the abbreviation $M_i = F_{\varepsilon_1} F_{\varepsilon_1 \varepsilon_2} \cdots F_{\varepsilon_1 \ldots \varepsilon_m}$; thus we are assuming $M_i M_i^* = A(.i + 2^{-m}) - A(.i)$. We would like to define F_{i0} positive so that (3) will hold for it, i.e., so that

$M_i F_{i0}^2 M_i^* = A(.i0+2^{-m-1}) - A(.i0)$. But the right-hand expression is a positive operator $\leqslant A(.i+2^{-m}) - A(.i)$, so by the Lemma we can do this. Next define F_{i1} as the positive square root of $1-F_{i0}^2$. It remains to prove (3) for $.i1$, that is, to prove that $M_i F_{i1}^2 M_i^* = A(.i1+2^{-m-1}) - A(.i1)$. By direct substitution, $M_i F_{i1}^2 M_i^* = A(.i+2^{-m}) - A(.i+2^{-m-1})$; so nothing more is needed but the obvious relation $.i1+2^{-m-1} = .i+2^{-m}$.

STEP 2 (Definition of the dilation). *Let $\mathcal{K}$ be the direct sum of countably many copies of $\mathcal{H}$, indexed by the dyadic rationals $.i=.\varepsilon_1\ldots\varepsilon_m$ $(m=0,1,\ldots)$. Embed $\mathcal{H}$ into $\mathcal{K}$ by ι, the natural mapping of $\mathcal{H}$ onto $\mathcal{H}_0$. Linear operators G_i from $\mathcal{K}$ to itself will be defined by the rule* (4) *below. Note that the notation* (1) *is still being used, so that j is a string of the same length as i. Note also that the $G_{\ldots}$ are indexed by strings, but the $\mathcal{H}_{\ldots}$ by rationals, so that $G_{j0} \neq G_j$ and yet $\mathcal{H}_{.j0} = \mathcal{H}_{.j}$. Thus it is all right for me to define $G_{i\nu}$ by telling how it acts on $\mathcal{H}_{.j0s} \oplus \mathcal{H}_{.j1s}$, because each component subspace of $\mathcal{K}$ is $\mathcal{H}_{.j0s}$ or $\mathcal{H}_{.j1s}$ for some j of length m and some s. The defintion is*

$$G_{i0} \quad \textit{transforms pairs} \begin{pmatrix} x \\ y \end{pmatrix} (x \in \mathcal{H}_{.j0s},\ y \in \mathcal{H}_{.j1s})$$

$$\textit{by the matrix} \begin{pmatrix} F_{i0} & 0 \\ F_{i1} & 0 \end{pmatrix}; \tag{4}$$

$$G_{i1}, \quad \textit{by the matrix} \begin{pmatrix} F_{i1} & 0 \\ -F_{i0} & 0 \end{pmatrix}.$$

Define operators B_i by $B_i = G_{i_1} G_{i_2} \cdots G_{i_{m-1}} G_i$ (again i_κ denotes the initial string of i of length κ). Finally, the operators $E(t)$ on $\mathcal{K}$ will be defined, for t a dyadic rational $\in [0,1]$, by starting with $E(0)=0$ and then requiring $E(.i+2^{-m}) - E(.i) = B_i B_i^*$.

It is evident that $E(t) \geqslant 0$. The definitions, with (3), do give $\iota^* B_i B_i^* \iota = A(.i+2^{-m}) - A(.i)$. In order to settle any fears that the last sentence of the statement of Step 2 might introduce an inconsistency, one needs to check that $B_{i0}B_{i0}^* + B_{i1}B_{i1}^* = B_i B_i^*$. But

$$\sum_{\nu=0}^{1} B_{i\nu} B_{i\nu}^* = G_{i_1} \cdots G_{i_{m-1}} G_i \left(\sum_{\nu=0}^{1} G_{i\nu} G_{i\nu}^* \right) G_i^* \cdots G_{i_1}^*,$$

and the sum in parentheses is 1 because on each subspace $\mathcal{H}_{.j0s} \oplus \mathcal{H}_{.j1s}$,

$G_{i\nu}G_{i\nu}^*$ has matrix

$$\begin{bmatrix} F_{i\nu}^2 & (-1)^{\nu}F_{i0}F_{i1} \\ (-1)^{\nu}F_{i0}F_{i1} & F_{i,\nu+1}^2 \end{bmatrix},$$

and these matrices add to 1. The special case of this where i is empty shows that $E(1)=1$ is consistent.

This has sufficed to define $E(\cdot)$ as an increasing positive-operator-valued function having the desired compression. To complete the proof it will be enough to prove that $B_iB_i^*B_jB_j^* = \delta_{ij}B_iB_i^*$. We have

$$B_i^*B_j = G_i^*G_{i_{m-1}}^* \cdots G_{i_1}^*G_{j_1} \cdots G_{j_{m-1}}G_j. \tag{5}$$

Now we compute from (4) that, for a string h of any length, $G_{h0}^*G_{h1} = G_{h1}^*G_{h0} = 0$, while $G_{h0}^*G_{h0} = G_{h1}^*G_{h1} =$ a direct sum of block matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Applying this (for h successively equal to $\varnothing$, $i_1,\ldots,i_{m-1}$) to (5), we see that $B_i^*B_jB_j^* = \delta_{ij}PB_i^*$ for a certain orthoprojector P; the reader may readily verify that $PB_i^* = B_i^*$.

Note that, except for notations, the last paragraph of this proof differed from that of Step 3 in Sec. 1 only in the occurrence of P, which in turn arose from the need to define the B_i on a direct sum of many copies of $\mathcal{H}$ instead of on a single copy.

The advantage claimed for this approach to Naĭmark's dilation theorem is, of course, that one may see the subspaces $E(t)\mathcal{K}$ lying alongside the subspace $\iota\mathcal{H} \subseteq \mathcal{K}$, and regard the operator $A(t)$ [or rather $\iota A(t)\iota^*$] as essentially the closeness operator [1] of those two subspaces: $\iota A(t)\iota^* = P_{\iota\mathcal{H}}E(t)P_{\iota\mathcal{H}}$. It has two disadvantages. First, the geometric simplicity is obfuscated by the need to index subspaces of $\mathcal{K}$ by the dyadic rationals. Second, the picture is changed unrecognizably by even an innocent reparametrization of the family $A(\cdot)$. Thus let the procedure be applied to $\{A(t^2): 0 \leqslant t \leqslant 1\}$, a family which satisfies the hypotheses and isn't much different from $A(\cdot)$. Then the picture will be simplest for $t=\sqrt{\frac{1}{2}}$; yet this, not being a dyadic rational, is not even one of the values of t for which the construction was explicit when $A(\cdot)$ was in question. In contrast, the usual proof of Naĭmark's theorem is altogether invariant under reasonable transformations of the argument.

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